

# HAUSDORFF DIMENSIONS FOR GRAPH-DIRECTED SELF-SIMILAR MEASURES DRIVEN BY INFINITE ROOTED TREES

KAZUKI OKAMURA

**ABSTRACT.** We give upper and lower bounds for Hausdorff dimensions for a class of graph-directed self-similar measures on a class of self-similar sets. Specifically, we consider the case that the underlying graph is the infinite  $N$ -ary tree. Our class contains self-similar measures, measures defined by restrictions of non-constant harmonic functions on the Sierpinski gasket, and, the energy measures on it, and, measures defined by solutions of de Rham's functional equations driven by  $N$  linear fractional transformations.

## CONTENTS

1. Introduction and Main results	1
2. Proof of Theorem 1.3	3
3. Examples	8
4. de Rham's functional equations driven by $N$ linear fractional transformations	14
5. Open problems	20
References	21

## 1. INTRODUCTION AND MAIN RESULTS

Let  $N \geq 2$ . Let  $\Sigma_N := \{0, 1, \dots, N-1\}$ . Let  $Y$  be a set and  $Z$  be a measurable space. For  $i \in \Sigma_N$ , let  $G_i : Y \rightarrow [0, 1]$  and  $H_i : Y \rightarrow Y$  be maps and  $F_i : Z \rightarrow Z$  be measurable maps. Then we consider the following equation for a family of probability measures  $\{\mu_y\}_{y \in Y}$  on  $Z$ :

$$\mu_y = \sum_{i=0}^{N-1} G_i(y) \mu_{H_i(y)} \circ F_i^{-1}, \quad (1.1)$$

under the assumption that

$$\sum_{i=0}^{N-1} G_i(y) = 1. \quad (1.2)$$

This family of measures can be regarded as the Markov type measures in Edgar-Mauldin [EM92], a self-similar family of measures in Strichartz [St93, Definition 2.2] and the graph directed self-similar measures in Olsen [Ol94, Section 1.1]. Let  $(V, E)$  be a directed graph which our framework concerns. (Here multi-edges and self-loops are allowed.) [EM92], [St93] and [Ol94] mainly focus on the case that  $V$  is finite. Several arguments in these references (for example, the Perron-Frobenius theorem

---

2010 *Mathematics Subject Classification.* 28A80, 60G30.

*Key words and phrases.* Hausdorff dimension for measure, graph-directed self-similar measures.

and ergodic theorem for finite Markov chains) depend on the fact that  $V$  is finite. Here we consider the case that  $(V, E)$  is the (suitably directed) *infinite*  $N$ -ary tree. See the definition of  $Y(y)$  below.

Our framework gives a unified approach for several objects in fractals which will look different at a glance. If  $V = Y(y) = \{y\}$ , then,  $\mu_y$  is a self-similar measure. However, in general,  $\mu_y$  may not be a self-similar measure. The class considered here also contains several measures appearing in fractals, specifically, self-similar measures, measures defined by solutions of de Rham's functional equations driven by linear fractional transformations, measures defined by restrictions of non-constant harmonic functions on the Sierpinski gasket, and, the energy measures on it. We will also discuss singularity of  $\mu_y$  with respect to self-similar measures.

We give upper and lower bounds for Hausdorff dimensions for  $\{\mu_y\}_y$  in the case that  $(Z, \{F_i\}_i)$  is a self-similar set on a complete metric space satisfying two conditions (See Assumption 1.2). As a result, we extend the main result of [ADF14]. Our proof will be an alternative proof of singularity of energy measures on the 2-dimensional Sierpinski gasket. Moreover, de Rham's functional equations driven by linear fractional transformations considered in [Ok14] are also generalized. Here we consider the functional equation driven by  $N$  functions, which is driven by only two functions in [Ok14]. A condition (A3) in [Ok14], which assures that the functions driving the equation are contractive, is loosed. Here we consider the functional equation driven by weak contractions.

Now we start to describe assumptions and main results. We first give notation and assumptions for  $Y$ . Let  $y \in Y$ . Let

$$Y(y) := \{H_{i_l} \circ \cdots \circ H_{i_1}(y) \mid i_1, \dots, i_l \in \Sigma_N, l \geq 1\}.$$

This corresponds to the infinite binary tree with root  $y$ . The set of edges is  $Y(y)$ , and the set of directed edges is  $\{(H_{i_l} \circ \cdots \circ H_{i_1}(y), H_{i_{l+1}} \circ \cdots \circ H_{i_1}(y)) \mid i_1, \dots, i_l, i_{l+1} \in \Sigma_N, l \geq 1\}$ , where we let  $H_{i_l} \circ \cdots \circ H_{i_1}(y) := y$  if  $l = 0$ .

**Assumption 1.1.** (i) We say that (A-y) holds if

$$0 < \inf_{i \in \Sigma_N, z \in Y(y)} G_i(z) \leq \sup_{i \in \Sigma_N, z \in Y(y)} G_i(z) < 1.$$

(ii) We say that (wA-y) holds if for some  $i \in \Sigma_N$  and  $c > 0$ ,

$$c \leq \inf_{z \in Y(y)} G_i(z) \leq \sup_{z \in Y(y)} G_i(z) \leq 1 - c.$$

(iii) We say that (B-y) holds if there exist  $\epsilon_0 \in (0, 1/N)$ ,  $l \geq 1$  and  $i_1, \dots, i_l \in \Sigma_N$  such that

$$\begin{aligned} Y(y) \cap \bigcap_{i \in \Sigma_N} G_i^{-1} \left( \left[ \frac{1}{N} - \epsilon_0, \frac{1}{N} + \epsilon_0 \right] \right) \\ \cap \bigcap_{j \in \Sigma_N} (G_j \circ H_{i_l} \circ \cdots \circ H_{i_1})^{-1} \left( \left[ \frac{1}{N} - \epsilon_0, \frac{1}{N} + \epsilon_0 \right] \right) = \emptyset. \end{aligned} \quad (1.3)$$

(iv) We say that (sB-y) holds if there exist  $\epsilon_0 \in (0, 1/N)$ ,  $l \geq 1$  such that (1.3) holds for any  $i_1, \dots, i_l \in \Sigma_N$ .

In the above we let  $Y$  as a set, but in order to check the conditions in Assumption 1.1, we will often put some topology on  $Y$ .

We second give notation and assumptions for  $Z$ . Let  $f_i, i \in \Sigma_N$ , be contractive maps on a complete metric space  $(M, d)$ , and  $K$  be the attractor of  $\{f_i\}_{i \in \Sigma_N}$ . We put

the Borel  $\sigma$ -algebra on  $K$  induced by the metric  $d$  on  $M$ . Let  $f_{i_1 \dots i_m} = f_{i_1} \circ \dots \circ f_{i_m}$  and  $K_{i_1 \dots i_m} = f_{i_1 \dots i_m}(K)$ .

**Assumption 1.2** (Cf. [Kig95, Corollary 1.3]). There exists constants  $r \in (0, 1)$ ,  $c_1, c_2 > 0$  and  $D > 0$  such that

- (i) For any  $m \geq 1$ ,  $\text{diam}(K_{i_1 \dots i_m}) \leq c_1 r^m$ .
- (ii) For any  $x \in K$  and  $m \geq 1$ ,  $|\{(i_1 \dots i_m) \in (\Sigma_N)^m \mid B(x, r^m) \cap K_{i_1 \dots i_m} \neq \emptyset\}| \leq D$ .

Let  $\dim_H A$  be the Hausdorff dimension of  $A$ . Let the Hausdorff dimension of  $\mu$  be  $\dim_H \mu := \inf\{\dim_H K \mid \mu(K) = 1\}$ .

**Theorem 1.3.** *Let  $Z = K$  and  $F_i = f_i$  satisfying Assumption 1.2. Then,*

- (i) *A family of solutions of (1.1), denoted by  $\{\mu_y\}_y$ , exists and is unique. Moreover,*

$$\sup_{y \in Y} \dim_H \mu_y \leq \frac{\log N}{\log(1/r)}. \quad (1.4)$$

- (ii) *Let  $y \in Y$ . Assume that both of (A-y) and (B-y) hold, or, only (sB-y) holds. Then,*

$$\dim_H \mu_y < \frac{\log N}{\log(1/r)}.$$

- (iii) *Let  $y \in Y$ . Assume that (wA-y) holds. Then,  $\dim_H \mu_y > 0$ .*

As an outline of our proof, we follow [Ok14, Theorem 1.2]. However, our method would be more transparent than [Ok14]. Specifically, we do not use four kinds of random subsets of natural numbers as in [Ok14, Lemma 3.3], which is the crucial part of the proof of [Ok14, Theorem 1.2]. See Lemma 2.6 for an alternative way. We emphasize that not only Theorem 1.3 is applicable to the models described above, but also there would be potential for applying it to different models. Indeed, we deal with some models other than the models described above. We also state some open problems.

The rest of the paper is organized as follows. Section 2 is devoted to the proof of Theorem 1.3. Section 3 is devoted to deal with various examples including the restriction of harmonic functions and the energy measures on the Sierpinski gasket. In Section 4, we deal with de Rham's functional equations. They are also examples, but we separate them from Section 3, because it takes a non-small portion of this paper. In Section 5, we state open problems.

## 2. PROOF OF THEOREM 1.3

**2.1. Proof of (i).** Let  $\mathbb{N} := \{1, 2, \dots\}$ . Here and henceforth we put the product  $\sigma$ -algebra on  $(\Sigma_N)^\mathbb{N}$  unless otherwise stated. Let  $\pi : (\Sigma_N)^\mathbb{N} \rightarrow K$  be a surjective map such that

$$\pi(ix) = f_i(\pi(x)) \text{ for any } i \in \Sigma_N \text{ and } x \in (\Sigma_N)^\mathbb{N}. \quad (2.1)$$

$\pi$  is uniquely determined.

For  $n \in \mathbb{N}$ , let  $X_n(x)$  be the projection of  $x \in (\Sigma_N)^\mathbb{N}$  to  $n$ -th coordinate. Let  $\mathcal{F}_n := \sigma(X_1, \dots, X_n)$ . This is a  $\sigma$ -algebra on  $(\Sigma_N)^\mathbb{N}$ . For  $n \geq 1$ , let

$$I(i_1, \dots, i_n) := \left\{ w \in (\Sigma_N)^\mathbb{N} \mid X_k(w) = i_k, 1 \leq k \leq n \right\}, \quad i_1, \dots, i_n \in \Sigma_N.$$

Consider the case that  $Z = (\Sigma_N)^\mathbb{N}$  and  $F_i(x) = ix$ .

By (1.2) and the Kolmogorov extension theorem, there exists a unique probability measure  $\nu_y$  on  $(\Sigma_N)^\mathbb{N}$  such that

$$\nu_y(I(i_1, \dots, i_n)) = \prod_{k=1}^n G_{i_k} \circ H_{i_{k-1}} \circ \dots \circ H_{i_1}(y).$$

Then,  $\{\nu_y\}_y$  is a family of solutions of (1.1). Then, by (2.1),  $\{\nu_y \circ \pi^{-1}\}_y$  is a family of solutions of (1.1) for  $Z = K$  and  $F_i = f_i$ .

Since each  $f_i$  is contractive, then, by following the proof of [F97, Theorem 2.8], we can show that a family of solutions of (1.1) for  $Z = K$  and  $F_i = f_i$  is unique. It follows that for any  $y$ ,

$$\nu_y \circ \pi^{-1} = \mu_y. \quad (2.2)$$

Let  $R_{y,n}(x) := \nu_y(I(X_1(x), \dots, X_n(x)))$ . By induction in  $n$ , it follows that

**Lemma 2.1.**

$$R_{y,n+1}(x) = R_{y,n}(x) G_{X_{n+1}(x)} \circ H_{X_n(x)} \circ \dots \circ H_{X_1(x)}(y).$$

Define  $P_N := \{(p_0, \dots, p_{N-1}) \in [0, 1]^N \mid \sum_{i \in \Sigma_N} p_i = 1\}$ . Define  $s_N : P_N \rightarrow \mathbb{R}$  by

$$s_N(p_0, \dots, p_{N-1}) := \sum_{i \in \Sigma_N} -p_i \log p_i.$$

Here we put  $0 \log 0 = 0$ . Then, by Jensen's inequality (for  $x \mapsto x \log x$ ),

$$s_N(p_0, \dots, p_{N-1}) \leq \log N, \quad (2.3)$$

and  $s_N(p_0, \dots, p_{N-1}) = \log N$  if and only if  $p_1 = \dots = p_N = 1/N$ .

It can happen that  $R_{y,n-1}(x) = 0$ , however,  $\nu_y(\{x \mid R_{y,n-1}(x) = 0\}) = 0$  holds for any  $n$ . Hence if we say “ $\nu_y$ -a.s. $x$ ”, then, we can assume that  $R_{y,n-1}(x) > 0$  for any  $n$ . Then, by Lemma 2.1,

**Lemma 2.2.**

$$E^{\nu_y} \left[ -\log \left( \frac{R_{y,n}}{R_{y,n-1}} \right) \mid \mathcal{F}_{n-1} \right] (x) = s_N \left( (G_j \circ H_{X_{n-1}(x)} \circ \dots \circ H_{X_1(x)}(y))_{j \in \Sigma_N} \right)$$

holds for  $\nu_y$ -a.s. $x$ .

Let  $M_{y,0} = 0$ . For  $n \geq 1$ , let

$$M_{y,n} - M_{y,n-1} := -\log \frac{R_{y,n}}{R_{y,n-1}} - E^{\nu_y} \left[ -\log \left( \frac{R_{y,n}}{R_{y,n-1}} \right) \mid \mathcal{F}_{n-1} \right].$$

(If  $R_{y,n-1}(x) = 0$ , then, we let  $(M_{y,n} - M_{y,n-1})(x) := 0$ . but such  $x$  does not affect integrations with respect to  $\nu_y$ .)

Then,  $\{M_{y,n}, \mathcal{F}_n\}_{n \geq 0}$  is a martingale under  $\nu_y$  and

**Lemma 2.3.**

$$\lim_{n \rightarrow \infty} \frac{M_{y,n}}{n} = 0, \quad \nu_y\text{-a.s.}$$

*Proof.* This part will be shown in the same manner as [Ok14, Lemma 2.3 (2)] by using Jensen's inequality and Doob's submartingale inequality. Let  $C := \sup_{(p_i)_{i \in P_N}} \sum_{i=0}^{N-1} p_i (-\log p_i)^2 < +\infty$ . Then, for any  $n \geq 1$ ,

$$E^{\nu_y} \left[ \left( \log \frac{R_{y,n+1}}{R_{y,n}} \right)^2 \right] \leq C.$$

By this and Jensen's inequality,  $\sup_{n \geq 0} E^{\nu_y} [(M_{y,n+1} - M_{y,n})^2] \leq 4C$ .

By Doob's submartingale inequality, for any  $\epsilon > 0$  and any  $n \geq 1$ ,

$$\begin{aligned} \nu_y \left( \max_{1 \leq k \leq 2^n} |M_{y,k}| \geq \epsilon 4^n \right) &\leq \frac{E^{\nu_y} [(M_{y,2^n})^2]}{\epsilon^2 4^{2n}} \\ &= \frac{\sum_{k \leq 2^n} E^{\nu_y} [(M_{y,k} - M_{y,k-1})^2]}{\epsilon^2 4^{2n}} \leq \frac{C}{\epsilon^2 4^{n-1}}. \end{aligned}$$

Therefore we have

$$\limsup_{n \rightarrow \infty} \frac{|M_{y,n}|}{n} \leq \sqrt{\epsilon}, \quad \nu_y\text{-a.s.}$$

□

For  $i \geq 1$ ,  $y \in Y$  and  $x \in (\Sigma_N)^\mathbb{N}$ , Let  $h_i(y; x) := H_{X_i(x)} \circ \cdots \circ H_{X_1(x)}(y)$ , and  $p_i(y; x) := (G_j \circ h_i(y, x))_{j \in \Sigma_N}$ .

By Lemmas 2.2 and 2.3,

**Corollary 2.4.** *For  $\nu_y$ -a.s.  $x$ ,*

$$\limsup_{n \rightarrow \infty} \frac{-\log R_{y,n}(x)}{n} = \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n s_N(p_{i-1}(y; x)),$$

and,

$$\liminf_{n \rightarrow \infty} \frac{-\log R_{y,n}(x)}{n} = \liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n s_N(p_{i-1}(y; x)).$$

**Lemma 2.5.** *If*

$$\limsup_{n \rightarrow \infty} \frac{-\log R_{y,n}}{n} \leq a, \quad \nu_y\text{-a.s.},$$

then, there exists a Borel subset  $B_0 \subset K$  such that  $\mu_y(B_0) = 1$  and

$$\dim_H(B_0) \leq \frac{a}{\log(1/r)}.$$

*Proof.* Let  $A_{n,\epsilon} := \{-\log R_n \leq n(a + \epsilon)\} \subset (\Sigma_N)^\mathbb{N}$ . Then, by the assumption and (2.2),

$$\mu_y \left( \pi \left( \bigcap_{k \geq 1} \bigcup_{m \geq 1} \bigcap_{n \geq m} A_{n,1/k} \right) \right) = 1.$$

Now it suffices to show that

$$\dim_H \pi \left( \bigcap_{k \geq 1} \bigcup_{m \geq 1} \bigcap_{n \geq m} A_{n,1/k} \right) \leq \frac{a}{\log(1/r)}. \quad (2.4)$$

Let

$$\mathcal{A}(n, s) := \{(i_1, \dots, i_n) \in (\Sigma_N)^n \mid \nu_y(I(i_1, \dots, i_n)) \geq \exp(-n(a + s))\}.$$

Since  $\sum_{(i_1, \dots, i_n) \in (\Sigma_N)^n} \nu_y(I(i_1, \dots, i_n)) = 1$ , we have that  $|\mathcal{A}(n, s)| \leq \exp(n(a + s))$ .

Fix  $k \geq 1$  and  $m \geq 1$ . By Assumption 1.2 (i),  $\text{diam}(\pi(I(i_1, \dots, i_n))) \leq c_1 r^n$ . Then, for any  $n \geq m$ ,

$$\pi \left( \bigcap_{n \geq m} A_{n,1/k} \right) \subset \bigcup_{(i_1, \dots, i_n) \in \mathcal{A}(n, 1/k)} \pi(I(i_1, \dots, i_n)),$$

and,

$$\begin{aligned} \sum_{(i_1, \dots, i_n) \in \mathcal{A}(n, 1/k)} \text{diam}(\pi(I(i_1, \dots, i_n)))^s &\leq c_1^s |\mathcal{A}(n, 1/k)| r^{sn} \\ &\leq c_1^s \exp\left(\left(a + \frac{1}{k} + s \log r\right) n\right). \end{aligned}$$

Hence,

$$\mathcal{H}_s\left(\pi\left(\bigcap_{n \geq m} A_{n, 1/k}\right)\right) = 0, \text{ if } s > \frac{a + 1/k}{\log(1/r)},$$

where we let  $\mathcal{H}_s$  be the  $s$ -dimensional Hausdorff measure on  $M$ . Hence,

$$\dim_H \pi\left(\bigcup_{m \geq 1} \bigcap_{n \geq m} A_{n, 1/k}\right) \leq \frac{a + 1/k}{\log(1/r)}.$$

Hence (2.4) follows.  $\square$

By this, Corollary 2.4 and (2.3), we have (1.4).

**2.2. Proof of (ii).** The following is different from a part of the proof of [Ok14, Theorem 1.2 (ii)], specifically, [Ok14, Lemma 3.3].

**Lemma 2.6.** *Assume that (1.3) holds for  $(i_k)_{1 \leq k \leq l}$ . Then, for any  $i \in \mathbb{N}$  and  $x \in (\Sigma_N)^\mathbb{N}$  satisfying  $X_{i+k}(x) = i_k, 1 \leq k \leq l$ ,*

$$\sum_{k=1}^l s_N(p_{i+k}(y; x)) \leq (l-1) \log N + \sup \left\{ s_N((p_j)_j) \left| \sum_j \left| p_j - \frac{1}{N} \right| > \epsilon_0 \right. \right\}.$$

If there exist  $\epsilon_0 \in (0, 1/N)$ ,  $l \geq 1$  such that (1.3) holds for any  $i_1, \dots, i_l \in \Sigma_N$ , then this inequality holds without the constraint that  $X_{i+k}(x) = i_k, 1 \leq k \leq l$ .

*Proof.* In this proof,  $\|\cdot\|$  denotes the  $\ell^1$ -norm on  $\mathbb{R}^l$ .

Case 1. Since

$$\begin{aligned} \left\| p_i(y; x) - \left( \frac{1}{N}, \dots, \frac{1}{N} \right) \right\| &> \epsilon_0, \\ s_N(p_i(y; x)) &\leq \sup \left\{ s_N((p_j)_j) \left| \sum_j \left| p_j - \frac{1}{N} \right| > \epsilon_0 \right. \right\}. \end{aligned}$$

By this and (2.3), it follows that

$$\sum_{k=1}^l s_N(p_{i+k}(y; x)) < (l-1) \log N + \sup \left\{ s_N((p_j)_j) \left| \sum_j \left| p_j - \frac{1}{N} \right| > \epsilon_0 \right. \right\}.$$

Case 2. If

$$\left\| p_i(y; x) - \left( \frac{1}{N}, \dots, \frac{1}{N} \right) \right\| \leq \epsilon_0,$$

and moreover  $X_{i+k}(x) = i_k, 1 \leq k \leq l$ , then, by (1.3),

$$\left\| p_{i+l}(y; x) - \left( \frac{1}{N}, \dots, \frac{1}{N} \right) \right\| \geq \left| G_{i+l}(h_{i+l}(y; x)) - \frac{1}{N} \right| > \epsilon_0,$$

and hence,

$$s_N(p_{i+l}(y; x)) \leq \sup \left\{ s_N((p_j)_j) \left| \sum_j \left| p_j - \frac{1}{N} \right| > \epsilon_0 \right. \right\}.$$

By this and (2.3), it follows that

$$\sum_{k=1}^l s_N(p_{i+k}(y; x)) < (l-1) \log N + \sup \left\{ s_N((p_j)_j) \left| \sum_j \left| p_j - \frac{1}{N} \right| > \epsilon_0 \right. \right\}.$$

Thus we have the assertion.  $\square$

**Lemma 2.7.** *For any  $i_1, \dots, i_l \in \Sigma_N$ , there exists a constant  $c_1 > 0$  such that  $\nu_y$ -a.s., there exists a random subset  $I(x) \subset \mathbb{N}$  such that*

$$\liminf_{n \rightarrow \infty} \frac{|I(x) \cap \{1, \dots, n\}|}{n} \geq c_1.$$

and for any  $i \in I(x)$ ,  $X_{i+k}(x) = i_k$ ,  $1 \leq k \leq l$ .

*Proof.* Fix  $i_1, \dots, i_l \in \Sigma_N$ . Let

$$\tilde{c} = \tilde{c}(y) := \inf_{i \in \Sigma_N, z \in Y(y)} G_i(z). \quad (2.5)$$

By (A-y),  $\tilde{c}(y) > 0$ . Let  $C(n) := \{X_{(n-1)l+k} = i_k, 1 \leq k \leq l\}$ . Let  $\widetilde{M}_0 := 0$  and for  $n \geq 1$ ,  $\widetilde{M}_n - \widetilde{M}_{n-1} := 1_{C(n)} - \tilde{c}^l$ . Then,  $|\widetilde{M}_n - \widetilde{M}_{n-1}| \leq 2$ , and,  $\{\widetilde{M}_n, \mathcal{F}_{ln}\}_n$  is a submartingale. Then, by Azuma's inequality [A67],

$$\nu_y \left( \sum_{k=1}^n 1_{C(k)} < \frac{n\tilde{c}^l}{2} \right) = \nu_y \left( \widetilde{M}_n < -\frac{n\tilde{c}^l}{2} \right) \leq \exp \left( -\frac{n\tilde{c}^{2l}}{32} \right).$$

Hence by the Borel-Cantelli lemma,

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n 1_{C(k)} \geq \frac{\tilde{c}^l}{2}, \quad \nu_y\text{-a.s.}$$

$\square$

*Proof of Theorem 1.3 (ii).* By Lemmas 2.6 and 2.7, there exists  $\epsilon_1 > 0$  such that

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n s_N(p_k(y; x)) \leq \log N - \epsilon_1, \quad \nu_y\text{-a.s. } x \in (\Sigma_N)^\mathbb{N}.$$

By this and Corollary 2.4,

$$\limsup_{n \rightarrow \infty} \frac{-\log R_{y,n}}{n} \leq \log N - \epsilon_1, \quad \nu_y\text{-a.s.}$$

By this and Lemma 2.5,

$$\dim_H \mu_y \leq \frac{\log N - \epsilon_1}{\log(1/r)}.$$

$\square$

### 2.3. Proof of (iii).

**Lemma 2.8.** *Assume*

$$\liminf_{n \rightarrow \infty} \frac{-\log R_{y,n}}{n} \geq a_2, \quad \nu_y\text{-a.s.}$$

Then,  $\mu_y(K_1) = 0$  holds for any Borel subset  $K_1$  of  $K$  such that

$$\dim_H(K_1) < \frac{a_2}{\log(1/r)}. \quad (2.6)$$

*Proof.* Let  $n \geq 1$  and  $\delta > 0$ . Assume (2.6) holds. Then, we can take open sets  $\{U_{n,l}\}_l$  in  $K$  such that for any  $l \geq 1$ ,  $\text{diam}(U_{n,l}) \leq c_2 r^n$ ,

$$K_1 \subset \bigcup_{l \geq 1} U_{n,l}, \quad (2.7)$$

and

$$\sum_{l \geq 1} \text{diam}(U_{n,l})^{a_2/\log(1/r)} < \delta. \quad (2.8)$$

Let  $k(n,l)$  be an integer such that  $c_2 r^{k(n,l)} \leq \text{diam}(U_{n,l}) < c_2 r^{k(n,l)-1}$ . Then, by  $k(n,l) \geq n$ , we have that

$$\nu_y(I_{k(n,l)}(x)) \leq \exp(-k(n,l)a_2) \leq \text{diam}(U_{n,l})^{a_2/\log(1/r)}$$

holds for any  $x \in \bigcap_{k \geq n} \{-\log R_{y,k} \geq ka_2\}$ . By this and Assumption 1.2 (ii),

$$\nu_y \left( \pi^{-1}(U_{n,l}) \cap \bigcap_{k \geq n} \{-\log R_{y,k} \geq ka_2\} \right) \leq D \text{diam}(U_{n,l})^{a_2/\log(1/r)}.$$

By this and (2.7) and (2.8),

$$\nu_y \left( \pi^{-1}(K_1) \cap \bigcap_{k \geq n} \{-\log R_{y,k} \geq ka_2\} \right) \leq D\delta.$$

By this, (2.2) and the assumption,  $\mu_y(K_1) = 0$ . □

By Lemma 2.8, Corollary 2.4, and (wA-y),

$$\dim_H \mu_y \geq \frac{-c \log c - (1-c) \log(1-c)}{\log(1/r)} > 0,$$

where  $c$  is the constant in (wA-y).

### 3. EXAMPLES

This section is devoted to state various examples, but we postpone de Rham's functional equations until the following section, because it takes longer pages than examples in this section.

#### 3.1. Self-similar sets satisfying Assumption 1.2 and self-similar measures.

As in [Kig95], the Sierpinski gasket, the Sierpinski carpet, the Koch curve, and the Lévy curve satisfy Assumption 1.2. Let  $V_m := \bigcup_{i_1, \dots, i_m \in \Sigma_N} f_{i_1, \dots, i_m}(V_0)$ . Let  $r_0 := \max_{i \in \Sigma_N} \text{Lip}(f_i)$ .

**Lemma 3.1.** *Assume that there exists  $D > 1$  such that for any large  $n$ ,*

$$\sup_{x \in V_n} |V_n \cap B(x, 2r_0^n \text{diam}(K))| \leq D. \quad (3.1)$$

*Then, Assumption 1.2 (ii) holds for  $r = r_0$ .*

*Proof.* Assume  $\text{diam}(U) \leq \text{diam}(K)r_0^n$ . Let  $x, y \in U$ . Then there exist points  $x_n, y_n \in V_n$  such that  $\max\{d(x, x_n), d(y, y_n)\} \leq r_0^n \text{diam}(K)$ . Then  $d(x_n, y_n) \leq 2r_0^n \text{diam}(K)$ . By the assumption, there are at most  $D$  sets of forms  $f_{i_1, \dots, i_n}(K)$  covering  $U$ . □



**Example 3.2.** We consider the Euclid metric.

- (i) If  $f_i(z) = (z + i)/N$ ,  $z \in \mathbb{R}$ , then,  $K = [0, 1]$ , and, Assumption 1.2 holds for  $r = 1/N$ .
- (ii) If  $N = 4$ ,  $f_i(z) = (z + q_i)/2$ ,  $z \in \mathbb{R}^2$ , where  $\{q_i\}_i = \{(x, y) \in \mathbb{Z}^2 \mid 0 \leq x, y \leq 1\}$ , then,  $K = [0, 1]^2$ , and, Assumption 1.2 holds for  $r = 1/2$ .
- (iii) If  $N = 3$ ,  $f_i(z) = (z + q_i)/2$ ,  $z \in \mathbb{R}^2$ , where  $\{q_i\}_i$  forms an equilateral triangle, then,  $K$  is a 2-dimensional Sierpinski gasket, and, Assumption 1.2 holds for  $r = 1/2$ .
- (iv) If  $N = 8$ ,  $f_i(z) = (z + q_i)/3$ ,  $z \in \mathbb{R}^2$ , where  $\{q_i\}_i = \{(x, y) \in \mathbb{Z}^2 \mid 0 \leq x, y \leq 2\} \setminus \{(1, 1)\}$ , then,  $K$  is a 2-dimensional Sierpinski carpet, and, Assumption 1.2 holds for  $r = 1/3$ .

*Proof.* Let  $L := \{\sum_i a_i q_i : a_i \in \mathbb{Z}\}$ . This is a discrete subset of  $[0, 1]$  or  $\mathbb{R}^2$ . By the definition of  $f_i$ , it follows that for any  $n$ ,  $V_n \subset N^{-n}L$ . Hence,  $\sup_{x, y \in V_n, x \neq y} d(x, y) \geq cN^{-n}$ , and, (3.1) holds for some  $D$ .  $\square$

Self-similar measures are introduced by Hutchinson [Hu81]. If  $Y$  is a one-point set, then, (1.1) does not depend on  $y$ , so we drop the notation. If (A) holds and  $(K, \Sigma_N, \{f_i\}_{i \in \Sigma_N})$  is a self-similar set, then, the solution  $\mu$  of (1.1) is a self-similar measure.

If Assumption 1.2 holds, then, by Corollary 2.4 and Lemmas 2.5, 2.8 and 3.1, we have that  $\dim_H \mu = s_N((G_j)_j) / \log(1/r)$ .

Hereafter, if  $(G_0, \dots, G_{N-1}) = (p_0, \dots, p_{N-1})$ , then, we call  $\nu$  and  $\mu$  the  $(p_0, \dots, p_{N-1})$ -Bernoulli measure and the  $(p_0, \dots, p_{N-1})$ -self-similar measure, respectively. We denote them by  $\nu_{(p_0, \dots, p_{N-1})}$  and  $\mu_{(p_0, \dots, p_{N-1})}$ , respectively.

### 3.2. Singularity with respect to self-similar measures.

**Proposition 3.3** (Singularity with respect to self-similar measures). *Let  $p_0, \dots, p_{N-1}$  be positive numbers satisfying  $\sum_{i \in \Sigma_N} p_i = 1$ . Assume that for some  $i_1, \dots, i_l$ ,*

$$\begin{aligned} Y(y) \cap \bigcap_{i \in \Sigma_N} G_i^{-1}([p_i - \epsilon_0, p_i + \epsilon_0]) \\ \cap \bigcap_{j \in \Sigma_N} (G_j \circ H_{i_l} \circ \dots \circ H_{i_1})^{-1}([p_j - \epsilon_0, p_j + \epsilon_0]) = \emptyset. \end{aligned} \quad (3.2)$$

Then,

- (i)  $\nu_y$  is singular with respect to  $\nu_{(p_0, \dots, p_{N-1})}$ .
- (ii) If moreover  $\pi^{-1}(\pi(A)) \setminus A$  is at most countable for any subset  $A$  of  $(\Sigma_N)^\mathbb{N}$ ,  $\mu_y$  is singular with respect to  $\mu_{(p_0, \dots, p_{N-1})}$ .

*Proof.* By Azuma's inequality,  $\nu_{p_0, \dots, p_{N-1}}$ -a.s.  $x$ , there are infinitely many  $i$  such that  $X_{i+k}(x) = i_k$  for any  $1 \leq k \leq l$ . By (3.2),  $\nu_{p_0, \dots, p_{N-1}}$ -a.s.  $x$ , there are infinitely many  $i$  such that

$$\|p_i(y; x) - (p_0, \dots, p_{N-1})\| \geq \epsilon_0.$$

Now assertion (i) follows from this and [Hi04, Theorem 4.1]. Assertion (ii) follows from (i), (2.2) and the assumption.  $\square$

**3.3. Energy measures on Sierpiński gaskets.** [Ku89] shows that energy measures for canonical Dirichlet forms on Sierpiński gaskets are singular with respect to the Hausdorff measure on them. It is generalized by [BST99], [Hi04]. Recently, [JOP] considers the Kusuoka measure from an ergodic theoretic viewpoint. Their framework covers a general class of measures that can be defined by products of matrices.

Let  $V_0 := \{q_0, q_1, q_2\}$  be the set of vertices of an equilateral triangle in  $\mathbb{R}^2$ . Let  $K$  be a 2-dimensional Sierpiński gasket, that is, the attractor of  $K = \cup_{i=0,1,2} f_i(K)$ , where we let  $f_i(z) := (z + q_i)/2, z \in \mathbb{R}^2$ .

Let  $a_i \in K, i = 0, 1, 2$ , be unique fixed points of  $F_i, i = 0, 1, 2$ , respectively. Let

$$A_0 := \begin{pmatrix} 3/5 & 0 \\ 0 & 1/5 \end{pmatrix}, A_1 := \begin{pmatrix} 3/10 & \sqrt{3}/10 \\ \sqrt{3}/10 & 1/2 \end{pmatrix}, A_2 := \begin{pmatrix} 3/10 & -\sqrt{3}/10 \\ -\sqrt{3}/10 & 1/2 \end{pmatrix}.$$

They are regular matrices and define linear transformation of  $Y$ .

Let  $\|\cdot\|$  be the Euclid norm on  $\mathbb{R}^2$ . Let  $Y := S^1 = \{x \in \mathbb{R}^2 : \|x\| = 1\}$ . We regard  $Y$  as a topological space with respect to the Euclid distance on  $Y \subset \mathbb{R}^2$ . For  $y \in Y$  and  $i = 0, 1, 2$ , let

$$G_i(y) := \frac{5}{3} \|A_i y\|^2, \text{ and, } H_i(y) := \frac{A_i y}{\|A_i y\|}.$$

**Lemma 3.4.** (i)  $A_0^2 + A_1^2 + A_2^2 = (3/5)I_2$ , where  $I_2$  denotes the identity matrix. In particular, (1.2) holds.

(ii) (A-y) holds for any  $y$ .

*Proof.* (i) is immediately seen. (ii) The set of eigenvalues of  $A_0, A_1, A_2$  are  $\{1/5, 3/5\}$ . Hence, for any  $i$  and  $y, 1/15 \leq G_i(y) \leq 3/5$ .  $\square$

**Lemma 3.5.** (i) If  $G_0(y) = G_0 \circ H_0(y)$ , then,  $y \in \{(\pm 1, 0), (0, \pm 1)\}$  and  $G_0(y) = G_0 \circ H_0(y) \in \{1/15, 3/5\}$ . In particular, if  $G_0(y) = 1/3$ , then,  $G_0 \circ H_0(y) \neq 1/3$ .

(ii) (B-y) holds for any  $y \in Y$ .

*Proof.* (i) is easy to see. Using (i) and that fact that  $G_i$  and  $H_i$  are continuous on  $Y$  and  $Y$  is compact, (B-y) follows.  $\square$

**Lemma 3.6.** Assume  $G_i(y) = p_i \in (0, 1), i = 0, 1, 2$ . Then,

(i) If  $y \notin \{(\pm 1, 0), (0, \pm 1)\}$ , then,  $G_0(H_0(y)) \neq p_0$ .

(ii) If  $y \in \{(\pm 1, 0), (0, \pm 1)\}$ , then,  $G_0(H_1(y)) \neq p_0$ .

Now we can apply Theorem 1.3 (ii), (iii) and Proposition 3.3 to this case, ( $N = 3, l = 2$ )

**Proposition 3.7.** It follows that  $0 < \dim_H \mu_y < \log 3 / \log 2$ . Moreover,  $\mu_y$  is singular with respect to any self-similar measure on  $K$ .

Let  $f$  be a harmonic function on  $K$ . Let  $h_1$  and  $h_2$  be the harmonic functions on  $K$  such that  $(h_1(q_0), h_1(q_1), h_1(q_2)) = (0, \sqrt{2}, \sqrt{2})$  and  $(h_2(q_0), h_2(q_1), h_2(q_2)) = (0, \sqrt{2/3}, -\sqrt{2/3})$ . Let  $v$  be the components of  $f$  in  $(h_1, h_2)$ . Let  $y = v/\|v\|$ . Then, the energy measure associated with  $f$  is  $\mu_y$ . (Cf. [BST99].)

**Remark 3.8.** (i) According to [JOP, Section 3], the Kusuoka measure is not a Gibbs measure, and therefore, it may not be applicable to the techniques of the thermodynamic formalism.

(ii) In a formal level, the framework adopted in [Hi04] is interpreted as follows. See [Hi04, Section 2] for details. Let  $(K, S, \{\psi_i\}_i)$  be a self-similar structure. Let  $\mu$  be a self-similar measure on  $K$ . Let  $(\mathcal{E}, \mathcal{F})$  be a regular Dirichlet form on  $L^2(K, \mu)$ . Assume (A1)-(A6) in [Hi04, Section 2].

$$Y := \mathcal{F},$$

$$G_i(f) := s_i \frac{\mathcal{E}(f \circ \psi_i, f \circ \psi_i)}{\mathcal{E}(f, f)}, \text{ if } \mathcal{E}(f, f) > 0.$$

$$G_i(f) := \frac{1}{|S|}, \quad \text{if } \mathcal{E}(f, f) = 0.$$

$$H_i(f) := f \circ \psi_i.$$

Then,  $\mu_f$  will be the normalized energy measure.

**3.4. Restriction of harmonic function on Sierpiński gasket.** [ADF14] considers the restriction on  $[0, 1]$  of harmonic functions on the Sierpiński gasket, and shows that they are singular functions<sup>1</sup> whenever they are monotone. The restrictions are among a wider class of functions containing several functions such as Lebesgue singular functions.

[ADF14, Notation 6] is interpreted as follows in our framework: Let

$$Y = [0, 1], \quad (3.3)$$

$$G_0(y) = \frac{2+y}{5}, \quad G_1(y) = 1 - G_0(y) = \frac{3-y}{5}, \quad (3.4)$$

and

$$H_0(y) = \frac{1+2y}{2+y}, \quad H_1(y) = \frac{y}{3-y}. \quad (3.5)$$

Then,  $f_y$  is the distribution function of  $\mu_y$ .

[ADF14, Theorem 3] is their main theorem, and, [ADF14, Lemma 24 and Theorem 25] restates it in a more general framework.

This is not of de Rham type in the above subsection, even if we exchange  $H_0$  with  $H_1$ . Theorem 1.3 (ii) gives the following improvements for [ADF14, Theorem 25]. (A-y) corresponds to [ADF14, assumption (a) in Lemma 24]. (sB-y) corresponds to [ADF14, assumption (b) in Lemma 24].

We loose the assumptions in two ways and simultaneously obtain a stronger conclusion. The first way for weakening the assumption is adopting (B-y), which is strictly weaker than [ADF14, assumption (b) in Lemma 24]. The second way is the case that (A-y) fails but (sB-y) holds.

**Theorem 3.9.** *Let  $Y = Z = [0, 1]$  and  $f_i(z) = (z + i)/N$ . Let  $y \in [0, 1]$ . Assume  $\mu_y$  has no atoms. Let  $f_y$  be the distribution function of  $\mu_y$ . Then,*

- (i) *If (A-y) and (B-y) hold, then,  $\dim_H \mu_y < 1$ .*
- (ii) *If (B-y) holds, then,  $f_y$  does not have non-zero derivative at almost every point with respect to any self-similar measure.*
- (iii) *If (sB-y) holds, then,  $f_y$  does not have non-zero derivative at any point, and,  $\dim_H \mu_y < 1$ .*

*Proof.* Assume that  $f_y$  has non-zero derivative at  $\pi(x) \in (0, 1)$ . Then,

$$\lim_{n \rightarrow \infty} G_{X_n(x)} \circ H_{X_{n-1}(x)} \circ \cdots \circ H_{X_1(x)}(y) = \frac{1}{N}. \quad (3.6)$$

Assume (B-y). Let  $\mu_{p_0, \dots, p_{N-1}}$  be a Bernoulli measure on  $\{0, 1\}^{\mathbb{N}}$ . Then by Azuma's inequality,  $\mu_{p_0, \dots, p_{N-1}} \left( \bigcap_{n \geq 1} \bigcup_{m \geq n} \bigcap_{k=1}^l \{X_{m+k} = i_k\} \right) = 1$ .

By this and (B-y), (3.6) fails for  $\mu_{p_0, \dots, p_{N-1}}$ -a.e.  $x \in \{0, 1\}^{\mathbb{N}}$ . Thus (ii) follows. (sB-y) implies that (3.6) fails for *any*  $x \in \{0, 1\}^{\mathbb{N}}$ .  $\square$

**Remark 3.10.** As one sees the proof, assertion (ii) above holds even if we replace any Bernoulli measure with any measure satisfying that there exists  $c \in (0, 1)$  such

---

<sup>1</sup>A singular function is a continuous, increasing function on  $[0, 1]$  whose derivative is zero almost surely with respect to the Lebesgue measure.

that for any  $m, k \geq 1$ ,

$$c\mu\left(\left[\frac{k-1}{2^m}, \frac{k}{2^m}\right]\right) \leq \mu\left(\left[\frac{k-1}{2^m}, \frac{2k-1}{2^{m+1}}\right]\right) \leq (1-c)\mu\left(\left[\frac{k-1}{2^m}, \frac{k}{2^m}\right]\right).$$

In the same manner as in the proof of Proposition 4.8,

**Proposition 3.11** (Singularity with respect to self-similar measures). *If (3.3), (3.4) and (3.5) hold, then,  $\mu_y$  is singular with respect to any self-similar measure.*

**Remark 3.12.** Our main results might be applicable to different models of Sierpinski gaskets such as level 3 Sierpinski gaskets. [ADF14] deals with the most canonical model (i.e. level 2) of Sierpinski gaskets.

**Remark 3.13.** The restrictions of harmonic functions on another model of self-similar sets have been considered. [ES16, Theorem 3.2] states that the restrictions of harmonic functions on the Hata tree is singular. Assume the framework of [ES16, Theorem 3.2]. Denote a restriction by  $f$ , and the measure whose distribution function is  $f$  by  $\mu_f$ . Then,

$$f(z) = \begin{cases} \frac{1}{|h|^2} f\left(\frac{z}{|\alpha|^2}\right) & 0 \leq z \leq |\alpha|^2, \\ \left(1 - \frac{1}{|h|^2}\right) f\left(\frac{z - |\alpha|^2}{1 - |\alpha|^2}\right) + \frac{1}{|h|^2} & |\alpha|^2 \leq z \leq 1, \end{cases} \quad (3.7)$$

This appears in the proof of [ES16, Theorem 3.2]<sup>2</sup>. See also Section 4 for functional equations of this kind.

Then,  $\mu_f$  satisfies (1.1) for  $Y = \{\text{one-point set}\}$ ,  $G_0 = |h|^{-2}$ ,  $Z = [0, 1]$ ,  $F_0(z) = |\alpha|^2 z$  and  $F_1(z) = (1 - |\alpha|^2)z + |\alpha|^2$ . (The notation  $F_1$  here is different from [ES16].) It is known that

$$\dim_H \mu_f = \frac{s_2(|h|^{-2}, 1 - |h|^{-2})}{-|h|^{-2} \log |\alpha|^2 - (1 - |h|^{-2}) \log(1 - |\alpha|^2)}.$$

Hence, if  $a \neq b$ , then,  $\dim_H \mu_f < 1$  and hence  $f$  is a singular function. Further, multifractal analysis for  $\mu_f$  is investigated. See [F97] for details.

### 3.5. Other examples.

**Example 3.14.** We give an example that (1.3) fails for any  $i_1, \dots, i_l$   $l = 1, 2$  both, but (1.3) holds for any  $i_1, \dots, i_l$   $l = 3$ . Let  $N = 2$ . Let  $Y = \mathbb{R}$ . Let

$$G_0(x) := \frac{1}{6} 1_{\{x < 0, x > 1\}} + \left(x + \frac{1}{6}\right) 1_{\{0 \leq x \leq 1/2\}} + \left(\frac{7}{6} - x\right) 1_{\{1/2 \leq x \leq 1\}}.$$

Let

$$H_0(y) = H_1(y) = \frac{5 - 3y}{6}.$$

Then,

$$\begin{aligned} G_0\left(\frac{1}{3}\right) &= G_0\left(H_0\left(\frac{1}{3}\right)\right) = G_0\left(\frac{2}{3}\right) = \frac{1}{2}, \\ G_0\left(H_0^2\left(\frac{1}{3}\right)\right) &= G_0\left(\frac{1}{2}\right) = \frac{2}{3}, \\ G_0\left(H_0^2\left(\frac{2}{3}\right)\right) &= G_0\left(H_0\left(\frac{1}{2}\right)\right) = G_0\left(\frac{7}{12}\right) = \frac{7}{12}. \end{aligned}$$

<sup>2</sup>There is a typo in it, and it will be fixed in (3.7).

**Example 3.15.** Let  $N = 2$  and  $Y = Z = [0, 1]$ . Let

$$G_0(x) = H_0(x) = \frac{1}{4}1_{\{x < 1/4\}} + x1_{\{1/4 \leq x \leq 3/4\}} + \frac{3}{4}1_{\{x > 3/4\}}.$$

Let  $H_1(x) = 1 - G_0(x)$ . Then,

$$G_0\left(\frac{1}{2}\right) = G_1\left(\frac{1}{2}\right) = H_0\left(\frac{1}{2}\right) = H_1\left(\frac{1}{2}\right) = \frac{1}{2},$$

and for any  $y \neq 1/2$ ,  $\dim_H \mu_y < 1$ . Therefore, if we did not introduce  $Y(y)$  in (b-y), then, the converse of Theorem 1.3 (ii) would not hold.

**Example 3.16.** Fix  $p \in (0, 1)$ . Let  $N = 2$  and

$$Y = [-\min\{p, 1-p\}^2, \min\{p, 1-p\}^2]$$

and  $Z = [0, 1]$ . Let

$$G_0(y) = p + \sqrt{|y|}, \text{ and } H_0(y) = H_1(y) = \frac{|y|}{|y| + 1}.$$

Then, for any  $x \in (\Sigma_N)^\mathbb{N}$  and  $y \in Y$ ,

$$\lim_{n \rightarrow \infty} G_0 \circ H_{X_n(x)} \circ \cdots \circ H_{X_1(x)}(y) = p.$$

By this and Corollary 2.4,  $\nu_y$ -a.s.  $x$ ,

$$\lim_{n \rightarrow \infty} \frac{-\log R_{y,n}(x)}{n} = \lim_{n \rightarrow \infty} s_2((G_j \circ H_{X_n(x)} \circ \cdots \circ H_{X_1(x)}(y))_j) = s_2(p).$$

By using Example 3.2 (i) and Lemmas 2.5 and 2.8, we can show that for any  $y$ ,  $\dim_H \mu_y = s_2(p)/\log 2$ .

Moreover  $\mu_y$  is the product measure on  $\{0, 1\}^\mathbb{N}$  such that

$$\mu_y(X_n = 0) = p + \sqrt{H_0^{n-1}(y)} = p + \sqrt{\frac{|y|}{(n-1)|y| + 1}}.$$

If  $y = 0$ , then,  $\mu_y$  is the Bernoulli measure with parameter  $(p, 1-p)$ . If  $y \neq 0$ , then, by Kakutani's dichotomy [Kak48],  $\mu_y$  is singular with respect to the Bernoulli measure.

**Example 3.17.** Let  $Z = [0, 1]$  and  $N = 2$ . Let  $Y = \mathbb{R}$ . Let

$$G_0(y) := \max\left\{\frac{1}{2} - |y|, 0\right\}, H_0(y) := (1 - \epsilon)y, \text{ and } H_1(y) := \epsilon^{-1}y.$$

Let  $y \neq 0$ . Obviously (A-y) fails. Then, by using that  $H_0$  is contractive and  $G_0(0) = G_1(0) = 1/2$  and 0 is the fixed points of  $H_0$  and  $H_1$  both, (B-y) fails.

We will show that  $\dim_H \mu_y < 1$ . For simplicity we assume  $y = 1/4$ . By Azuma's inequality, if we take sufficiently small  $\epsilon > 0$ , then, there exists  $D > 1$  such that for any  $n \geq 1$ ,

$$\ell\left(\left\{x \in \{0, 1\}^\mathbb{N} \mid |H_{X_n(x)} \circ \cdots \circ H_{X_1(x)}(1/4)| < 1/2\right\}\right) \leq D^{-n},$$

where  $\ell$  is the  $(1/2, 1/2)$ -Bernoulli measure on  $\{0, 1\}^\mathbb{N}$ . Therefore, by the definition of  $\nu_y$ , for any  $n$ ,

$$\begin{aligned} & |\{(i_1, \dots, i_n) \in \{0, 1\}^n \mid \nu_y(I(i_1, \dots, i_n, 0)) > 0\}| \\ &= |\{(i_1, \dots, i_n) \in \{0, 1\}^n \mid |H_{i_n} \circ \cdots \circ H_{i_1}(1/4)| < 1/2\}| \leq \left(\frac{2}{D}\right)^n. \end{aligned}$$

Hence, by taking sum with respect to  $n$ ,

$$\begin{aligned} & |\{(i_1, \dots, i_m) \in \{0, 1\}^m \mid \nu_y(I(i_1, \dots, i_m)) > 0\}| \\ &= \sum_{n < m} |\{(i_1, \dots, i_n) \in \{0, 1\}^n \mid \nu_y(I(i_1, \dots, i_n, 0, 1, \dots, 1)) > 0\}| \\ &\leq \sum_{n < m} \left(\frac{2}{D}\right)^n \leq C \left(\frac{2}{D}\right)^m. \end{aligned}$$

By this and  $D > 1$ ,

$$\dim_H \mu_{1/4} < 1.$$

**Example 3.18.** Let  $Z = [0, 1]$  and  $N = 2$  and  $f_i(z) = (z + i)/2$ . It is easy to construct an example such that  $\mu_y$  is not absolutely continuous or singular with respect to a self-similar measure  $\mu_{(p, 1-p)}$ . Let  $Y = \{0, 1, 2\}$ . Let  $H_0(0) = 1$ ,  $H_1(0) = 2$ , and  $H_i(j) = j$ ,  $j = 1, 2$ ,  $i = 0, 1$ . Let  $G_0(j) = 1/2$ ,  $j = 1, 2$ . Let  $\mu_1 = \mu_{(p, 1-p)}$  and  $\mu_2$  be a probability measure which is singular with respect to  $\mu_{(p, 1-p)}$ . Then, by the uniqueness of the Lebesgue decomposition,  $\mu_0$  is not absolutely continuous or singular with respect to  $\mu_{(p, 1-p)}$ .

#### 4. DE RHAM'S FUNCTIONAL EQUATIONS DRIVEN BY $N$ LINEAR FRACTIONAL TRANSFORMATIONS

Before we apply our results to a class of de Rham's functional equations, we give a short review.

**4.1. Short introduction for de Rham's functional equations.** De Rham [dR56, dR57]<sup>3</sup> considered a certain class of functional equations. He considered the solution  $f$  of the following functional equation which takes its values in a certain metric space:

$$f(x) = \begin{cases} J_0(f(Nx)) & 0 \leq x \leq 1/N, \\ J_1(f(Nx - 1)) & 1/N \leq x \leq 2/N, \\ \dots & \\ J_{N-1}(f(Nx - (N-1))) & (N-1)/N \leq x \leq 1, \end{cases} \quad (4.1)$$

where each  $J_i$  is a weak contraction,  $J_0(0) = 0$ ,  $J_{N-1}(1) = 1$  and  $J_{i+1}(0) = J_i(1)$  for each  $i$ . Solutions of de Rham's functional equations give parameterizations of some self-similar sets such as the Kôch curve and the Pólya curve, etc. Some singular functions such as the Cantor, Lebesgue, etc. functions are solutions of such functional equations.

We give a short review of some known results. [BK00] considers self-similarity, inversion and composition of de Rham's functions, and points out a connection with Collatz's problem. [Kr09] shows connections between sums related to the binary sum-of-digits function and the Lebesgue's singular function, and its partial derivatives with respect to the parameter. [Kaw11] investigates the set of points where Lebesgue's singular function has the derivative zero. [P04] regards a de Rham curve as the limit of a polygonal arc by repeatedly cutting off the corners, obtain a formula for the local Hölder exponent of a de Rham curve at each point, and describe the sets of points with given local regularity. Recently, [BKK] performs the multifractal analysis for the pointwise Hölder exponents of zipper fractal curves generated by affine mappings. [BKK] and [N04] consider the Hausdorff dimension

<sup>3</sup>An English translation of [dR57] is included in Edgar [E93].

of the image measure of the Lebesgue measure on an interval by the de Rham function. [PV17, Section 9.3] also consider de Rham curves in terms of matrix products. [DL91, DL92-1, DL92-2, P06] are related to wavelet theory. The length of de Rham curve is investigated in [Me98] and [DMS98]. [TGD98] uses de Rham's functional equations to construct the Sinai-Ruelle-Bowen measures for several Baker-type maps. de Rham's functional equations also appear in [DS76], which is concerned with gambling strategy. [G93, Z01, SB17] consider generalizations of de Rham's functions.

The case that some of  $J_i$  are not affine is considered in [Ha85, SLK04, Ok14, Ok16], however, it seems that the case is not fully investigated. [SLK04] considers it from a view point of random walk. [Ha85] and [SLK04] use the technique of [La73].

**4.2. de Rham's functional equations driven by  $N$  linear fractional transformations.** [Ok14] considers de Rham's functional equations driven by *two* linear fractional transformations, here we consider not only the case that  $N = 2$  and but also the case that  $N \geq 3$ . Our outline is similar to the one in [Ok14], but several additional considerations are needed.

Let  $(M, d)$  be a metric space. Following [Ha85, Definition 2.1], we say that a function  $f : M \rightarrow M$  is a *weak contraction* if for any  $t > 0$ ,

$$\lim_{s \rightarrow t, s > t} \sup_{d(x, y) \leq \delta} d(f(x), f(y)) < t.$$

By [Ha85, Corollary 6.6], if each  $J_i$  is a weak contraction,  $J_0(0) = 0$ ,  $J_{N-1}(1) = 1$  and  $J_{i+1}(0) = J_i(1)$  for each  $i$ , then, (4.1) has a unique continuous solution  $f$  and we let  $\mu = \mu_f$  be the probability measure such that the solution  $f$  of (4.1) is the distribution function of  $\mu_f$ .

In the following, we consider the equation (4.1) for the case that all  $J_i$  are *linear fractional transformations*. Let  $\Phi(A; z) := \frac{az + b}{cz + d}$  for a  $2 \times 2$  real matrix  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  and  $z \in \mathbb{R}$ . Let

$$J_i(x) := \Phi(A_i; x), \quad x \in [0, 1], \quad i = 0, 1, \dots, N-1,$$

such that  $2 \times 2$  real matrices  $A_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$ ,  $i = 0, 1$ , satisfy the following conditions

(A1) - (A3).

(A1)  $\Phi(A_0; 0) = 0$ ,  $\Phi(A_{N-1}; 1) = 1$ , and

$$\Phi(A_{i-1}; 1) = \Phi(A_i; 0), \quad i = 1, \dots, N-1.$$

(A2)

$$\det A_i = a_i d_i - b_i c_i > 0, \quad i = 0, 1, \dots, N-1.$$

(A3)

$$(a_i d_i - b_i c_i)^{1/2} \leq \min\{d_i, c_i + d_i\}, \quad i = 0, 1, \dots, N-1.$$

In several cases, we will replace (A3) with a stronger condition (sA3):

$$(a_i d_i - b_i c_i)^{1/2} < \min\{d_i, c_i + d_i\}, \quad i = 0, 1, \dots, N-1.$$

By (A2) and (A3),  $d_i > 0$ , and henceforth, we can assume  $d_i = 1$  for any  $i$ , without loss of generality. By (A1),  $b_0 = 0$ , and,  $0 < b_i < 1$ ,  $1 \leq i \leq N-1$ . By (A3),  $a_0 \leq 1$ .

**Lemma 4.1.** *For each  $i$ ,  $\Phi(A_i; x)$  is a weak contraction on  $[0, 1]$ .*

*Proof.* If  $c_i = 0$ , then,  $\Phi(A_i; x) = a_i x + b_i$ . By (A2)  $a_1 > 0$ . By (A1),  $a_i + b_i = b_{i+1} < 1$  and  $b_i \geq 0$  for  $i \leq N-2$ , and,  $a_i + b_i \leq 1$  and  $b_i > 0$  for  $i \geq 1$ . Hence  $a_i < 1$ . Thus  $\Phi(A_i; x)$  is a contraction on  $[0, 1]$ .

If  $c_i \neq 0$ , then, by (A3), for any  $t > 0$ ,

$$\min_{t < |x-y|} \frac{a_i - b_i c_i}{(c_i x + 1)(c_i y + 1)} < 1.$$

Hence,  $\Phi(A_i; x)$  is a weak contraction.  $\square$

Therefore, (A1) - (A3) guarantee that (4.1) has a unique continuous solution  $f$ . The above (sA3) is identical with (A3) in [Ok14].

We remark that our framework contains the cases that the technique of [La73] appearing in [Ha85, Theorem 7.3] and [SLK04, Proposition 3.1] would not be applicable. By (A1) - (A3),  $b_i - c_i \leq 1 - a_i \leq 1 - b_i c_i$ . Hence,

$$(1 - a_i)^2 + 4b_i c_i \geq \min\{(b_i + c_i)^2, (1 + b_i c_i)^2\} \geq 0.$$

Let

$$\alpha := \min \left\{ 0, \frac{c_0}{1 - a_0}, \frac{a_i - 1 + \sqrt{(1 - a_i)^2 + 4b_i c_i}}{2b_i} \mid 1 \leq i \leq N-1 \right\}.$$

If  $a_0 = 1$ , then, we replace  $c_0/(1 - a_0)$  with  $-1$ .

Let

$$\beta := \max \left\{ 0, \frac{c_0}{1 - a_0}, \frac{a_i - 1 + \sqrt{(1 - a_i)^2 + 4b_i c_i}}{2b_i} \mid 1 \leq i \leq N-1 \right\}.$$

If  $a_0 = 1$ , then, we replace  $c_0/(1 - a_0)$  with  $+\infty$ .

Let  $Y := [\alpha, \beta]$ . We consider the topology of  $Y$  defined by the Euclid metric. For  $k \in \Sigma_N$  and  $y \in Y$ , let

$$G_k(y) := \frac{(a_k - b_k c_k)(y + 1)}{(b_k y + 1)((a_k + b_k)y + c_k + 1)}, \text{ and } H_k(y) := \frac{a_k y + c_k}{b_k y + 1}.$$

If  $a_0 = 1$ , then,  $G_0(+\infty) := 1, G_k(+\infty) := 0, H_0(+\infty) := +\infty, H_k(+\infty) := \frac{a_k}{b_k}, 1 \leq k \leq N-1$ .

**Lemma 4.2.** (i) If  $a_0 = 1$ , then,  $\alpha = -1$  and  $\beta = +\infty$ . If  $a_0 < 1$  and  $b_{N-1} + c_{N-1} = 0$ , then,  $-1 = \alpha \leq \beta < +\infty$ . If  $a_0 < 1$  and  $b_{N-1} + c_{N-1} > 0$ , then,  $-1 < \alpha \leq \beta < +\infty$ .

(ii)  $H_i(y) \in Y$  for  $i \in \Sigma_N, y \in Y$ .

(iii) (1.2) holds.

*Proof.* (i) By (A1), it follows that if  $1 \leq i \leq N-2$ , then,

$$\frac{a_i - 1 - \sqrt{(1 - a_i)^2 + 4b_i c_i}}{2b_i} < -1 < \frac{a_i - 1 + \sqrt{(1 - a_i)^2 + 4b_i c_i}}{2b_i}. \quad (4.2)$$

By (A2) and (A3),  $b_{N-1} + c_{N-1} \geq 0$ . By this and (A1), it follows that if  $i = N-1$ , then,

$$\frac{a_{N-1} - 1 - \sqrt{(1 - a_{N-1})^2 + 4b_{N-1}c_{N-1}}}{2b_{N-1}} = -1, \quad (4.3)$$

and,

$$\leq \frac{c_{N-1}}{b_{N-1}} = \frac{a_{N-1} - 1 + \sqrt{(1 - a_{N-1})^2 + 4b_{N-1}c_{N-1}}}{2b_{N-1}}. \quad (4.4)$$



Hence, if  $a_0 = 1$ , then,  $\alpha = -1$ . If  $a_0 < 1$ , then, by (A1),  $a_0 < c_0 + 1$ . Hence, if  $a_0 < 1$  and  $b_{N-1} + c_{N-1} > 0$ , then,  $\alpha > -1$ .

(ii) Let  $i = 0$ . First we remark that  $H_0(z) = a_0 z + c_0$ . Assume  $a_0 < 1$ . Then, by  $\alpha \leq c_0/(1 - a_0) \leq \beta$ , we see that  $\alpha \leq H_0(\alpha) \leq H_0(\beta) \leq \beta$ . If  $a_0 = 1$ , then,  $\beta = +\infty$ . It is easy to see that  $\alpha \leq H_0(\alpha)$ .

Let  $1 \leq i \leq N - 1$ . Then,  $H_i(z) \geq z$  if and only if

$$\frac{a_i - 1 - \sqrt{(1 - a_i)^2 + 4b_i c_i}}{2b_i} \leq z \leq \frac{a_i - 1 + \sqrt{(1 - a_i)^2 + 4b_i c_i}}{2b_i}. \quad (4.5)$$

Hence,  $H_i(\beta) \leq \beta$ .

By (A2), for any  $k$ ,  $H_k(z)$  is monotone increasing on  $z \geq -1$ . By this, (4.2), (4.3), (4.4), and (4.5),  $H_i(\alpha) \geq \alpha$ .

(iii) It is easy to see by calculations that for any  $i \geq 0$ ,

$$G_i(y) = \frac{(y + 1)(b_{i+1} - b_i)}{(b_{i+1}y + 1)(b_iy + 1)}.$$

The assertion follows from this and (A1).  $\square$

**Lemma 4.3.** *If (sA3) holds, then, (A-y) holds for  $y = 0$ .*

*Proof.* By (sA3),  $1 > a_0$  and  $b_{N-1} + c_{N-1} > 0$ . By this and Lemma 4.2 (i), we have that  $-1 < \alpha \leq \beta < +\infty$ . Therefore,  $0 < G_0(\alpha) \leq G_0(\beta) < 1$ .

Let  $i \geq 1$ . Since  $\alpha > -1$  and  $0 < b_i < 1$ ,  $\inf_{y \geq \alpha} G_i(y) > 0$ . By the definition of  $G_i$ , we can show that if  $y \geq -1$ , then,  $G_i(y) < 1$ . Hence, by continuity of  $G_i$ ,  $\sup_{y \in [\alpha, \beta]} G_i(y) < 1$ .  $\square$

Hereafter we denote the set of fixed points of  $f$  by  $\text{Fix}(f)$ .

**Remark 4.4.** (i) In the above proof, we have used  $b_{N-1} + c_{N-1} > 0$ . However, if  $i < N - 1$ , then,  $b_i + c_i > 0$  may fail.

(ii) If  $i \geq 1$ ,  $G_i$  may not be increasing.

(iii)  $\left| \bigcap_{i=0}^{N-1} \text{Fix}(H_i) \right| \leq |\text{Fix}(H_0)| = 1$ . If  $a_0 < 1$ , then,  $\text{Fix}(H_0) = \{c_0/(1 - a_0)\}$ . If  $a_0 = 1$ , then, by extending the domain of  $H_0$ ,  $\text{Fix}(H_0) = \{+\infty\}$ .

**Lemma 4.5.** *Assume  $\{\nu_y\}_{y \in Y}$  satisfies (1.1) for  $Z = (\Sigma_N)^{\mathbb{N}}$  and  $F_i(x) = ix$ . Then,  $\pi(x) = \sum_{i \geq 1} X_i(x)/N^i$ , and,  $\mu_f = \nu_0 \circ \pi^{-1}$ .*

*Proof.* Let

$$\begin{pmatrix} p_n(x) & q_n(x) \\ r_n(x) & s_n(x) \end{pmatrix} := A_{X_1(x)} \cdots A_{X_n(x)}, \quad n \geq 1.$$

We have that

$$\begin{aligned} \mu_f(\pi(I_k(x))) &= \mu_f\left(\left[\sum_{j=1}^k \frac{X_j(x)}{2^j}, \sum_{j=1}^k \frac{X_j(x)}{2^j} + \frac{1}{2^k}\right]\right) \\ &= \Phi(A_{X_1(x)} \cdots A_{X_k(x)}; 1) - \Phi(A_{X_1(x)} \cdots A_{X_k(x)}; 0) \\ &= \frac{p_k(x)s_k(x) - q_k(x)r_k(x)}{s_k(x)(r_k(x) + s_k(x))}. \end{aligned}$$

Therefore by computation,

$$\frac{\mu_f(\pi(I_{n+1}(x)))}{\mu_f(\pi(I_n(x)))} = \frac{R_{0,n+1}(x)}{R_{0,n}(x)} = \frac{\nu_0(I_{n+1}(x))}{\nu_0(I_n(x))}.$$

Hence  $\mu_f(\pi(I_n(x))) = \nu_0(I_n(x))$ . Since  $f$  is continuous, we have that  $\lim_{n \rightarrow \infty} \mu_f(\pi(I_n(x))) = 0$ , and hence,  $\nu_0$  has no atoms. Since  $\pi^{-1}(\pi(I_n(x))) \setminus I_n(x)$  is at most countable,

$\nu_0(\pi^{-1}(\pi(I_n(x))) \setminus I_n(x)) = 0$ . Since  $\{\pi(I_n(x)) \mid n \geq 1, x \in (\Sigma_N)^{\mathbb{N}}\}$  generates the Borel  $\sigma$ -algebra on  $[0, 1]$ ,  $\nu_0 \circ \pi^{-1} = \mu_f$ .  $\square$

**Theorem 4.6** (Upper bound for Hausdorff dimension). *(i) If either*

$$\bigcap_{i=0}^{N-1} G_i^{-1} \left( \frac{1}{N} \right) \neq \emptyset \quad (4.6)$$

*or*

$$\bigcap_{i=0}^{N-1} G_i^{-1} \left( \frac{1}{N} \right) \subset \bigcap_{i=0}^{N-1} \text{Fix}(H_i) \quad (4.7)$$

*fails, then,*

$$\dim_H \mu < 1.$$

*(ii) If (4.6) and (4.7) hold, then, the distribution function  $\mu$  is given by*

$$f(x) = \frac{x}{1 - C_{N,0}(x-1)}, \text{ where we let } C_{N,0} := \frac{c_0 N}{N-1}. \quad (4.8)$$

In other words, if the solution  $f$  of (4.1) is not of the form of (4.8), then,  $\dim_H \mu_f < 1$ , and hence,  $f$  is a singular function.

*Proof.* (i) If (4.6) fails and  $\beta < +\infty$ , then, by the continuity of  $G_i$  and the compactness of  $Y$ ,  $\inf_{y \in Y} \sum_{i \in \Sigma_N} |G_i(y) - N^{-1}| > 0$ . This holds even when  $\beta = +\infty$ , by recalling the definition of  $G_i(+\infty)$ . Hence,

$$\sup_{x \in (\Sigma_N)^{\mathbb{N}}, y \in Y, i \in \mathbb{N}} s_N(p_i(y; x)) < \log N.$$

Now the assertion follows from Corollary 2.4 and Lemma 2.5.

Assume that (4.6) holds and (4.7) fails. Using (4.6) and

$$\begin{aligned} G_0^{-1} \left( \frac{1}{N} \right) &= \left\{ \frac{1 + c_0 - a_0 N}{a_0(N-1)} \right\}, \\ \bigcap_{i=0}^{N-1} G_i^{-1} \left( \frac{1}{N} \right) &= \left\{ \frac{1 + c_0 - a_0 N}{a_0(N-1)} \right\}. \end{aligned} \quad (4.9)$$

Since (4.7) fails, for some  $i$ ,

$$H_i \left( \frac{1 + c_0 - a_0 N}{a_0(N-1)} \right) \neq \frac{1 + c_0 - a_0 N}{a_0(N-1)}.$$

Since  $G_0$  is monotone increasing, (1.3) holds for  $i_1 = i$ .

(ii) By the assumption, (4.9) holds, and hence it follows that

$$G_i \left( \frac{1 + c_0 - a_0 N}{a_0(N-1)} \right) = \frac{1}{N},$$

and

$$H_i \left( \frac{1 + c_0 - a_0 N}{a_0(N-1)} \right) = \frac{1 + c_0 - a_0 N}{a_0(N-1)}.$$

By using them, we see that for any  $i \in \Sigma_N$ ,

$$a_i = \frac{(N+1)C_{N,0}b_i + 1}{N}, \text{ and } c_i = \frac{C_{N,0}(N-1 - C_{N,0}b_i)}{N}. \quad (4.10)$$

We now show that

$$b_i = \frac{i}{(N-i)C_{N,0} + N}. \quad (4.11)$$

by induction in  $i$ . We remark that by (A2) and (A3),  $c_0 + 1 > 0$ , and hence  $(N - i)C_{N,0} + N > 0$  for any  $i \leq N - 1$ .

If  $i = 0$ , then, by (A1),  $b_0 = 0$ . Assume that (4.11) holds for  $i = k$ . By (4.10), (4.11), and (A1),

$$b_{k+1} = \frac{k+1}{(N-k-1)C_{N,0} + N}.$$

Now it is easy to see that  $f$  given by (4.8) satisfies (4.1).  $\square$

**Remark 4.7** (Lower bound for Hausdorff dimension). Assume  $a_0 < 1$  and  $b_{N-1} + c_{N-1} > 0$ . Let  $\tilde{c}$  be the constant in (2.5). Then, by Corollary 2.4 and Lemma 2.8,

$$\dim_H \mu_f \geq \frac{\inf \{s_N((p_j)_j) \mid (p_j)_j \in P_N, \tilde{c} \leq p_j \leq 1 - \tilde{c}, \forall j\}}{\log N} > 0.$$

**Proposition 4.8** (Singularity with respect to self-similar measures). *Let  $p_i \in (0, 1)$ ,  $0 \leq i \leq N - 1$ . Let  $C_0 := c_0/(1 - a_0)$  and assume  $a_0 < 1$ . Then,*

(i) *If there exist  $p_i \in (0, 1)$ ,  $0 \leq i \leq N - 1$  such that*

$$a_i = \frac{p_i + C_0 \sum_{j=0}^i p_j}{\left(1 - \sum_{j=0}^{i-1} p_j\right) C_0 + 1}. \quad (4.12)$$

$$b_i = \frac{\sum_{j=0}^{i-1} p_j}{\left(1 - \sum_{j=0}^{i-1} p_j\right) C_0 + 1}. \quad (4.13)$$

$$c_i = \frac{C_0 \left( \left(1 - \sum_{j=0}^i p_j\right) C_0 + 1 - p_i \right)}{\left(1 - \sum_{j=0}^{i-1} p_j\right) C_0 + 1}. \quad (4.14)$$

*hold for any  $i$ , then,  $\mu_f$  is absolutely continuous with  $\mu_{(p_0, \dots, p_{N-1})}$ .*

(ii) *Otherwise,  $\mu_f$  is singular with respect to any self-similar measure.*

We do not know about an explicit expression for the Radon-Nikodym derivative  $d\mu_f/d\mu_{(p_0, \dots, p_{N-1})}$ .

*Proof.* (i) We first remark that in this case,  $0 < a_0 = p_0 < 1$  and  $\beta < +\infty$ . By computation,  $H_i(C_0) = C_0$ , and  $H'_i(C_0) = p_i$ , where  $H'_i$  denotes the derivative of  $H_i$ .

Hence, each  $H_i$  is contractive on a neighborhood  $U$  of  $C_0$ . Since  $H_0$  is contractive on  $[\alpha, \beta]$ , there exists  $M$  such that for any  $x \in [\alpha, \beta]$ ,  $H_0^M(x) \in U$ .

By Azuma's inequality,  $\nu_0 \left( \bigcup_{i \geq 1} \bigcap_{j=1}^M \{X_{i+j} = 0\} \right) = 1$ . Hence,  $\nu_0$ -a.s.  $x$ ,  $\lim_{n \rightarrow \infty} H_{X_n(x)} \circ \dots \circ H_{X_1(x)}(0) = C_0$ , and this convergence is exponentially fast<sup>4</sup>. Hence,  $\nu_0$ -a.s.  $x$ ,

$$\sum_{n \geq 1} 1 - \sum_{i \in \Sigma_N} \sqrt{p_i G_i \left( H_{X_n(x)} \circ \dots \circ H_{X_1(x)}(0) \right)} < +\infty.$$

By this and [Sh80, Theorem VII.6.4],  $\nu_0$  is absolutely continuous with respect to  $\nu_{(p_0, \dots, p_{N-1})}$ . Hence,  $\mu_f$  is absolutely continuous with respect to  $\mu_{(p_0, \dots, p_{N-1})}$ .

(ii) Let  $\mu_{(p_0, \dots, p_{N-1})}$  be the  $(p_0, \dots, p_{N-1})$ -self-similar measure. By the definition of  $\pi$ , we have that  $\pi^{-1}(\pi(A)) \setminus A$  is at most countable for any  $A$ . First we consider the case that  $p_0 \neq a_0$ . Then,  $\text{Fix}(H_0) = C_0 \neq G_0^{-1}(p_0)$ . Therefore, if  $G_0(y) = p_0$ , then,  $G_0(H_0(y)) \neq p_0$ . Thus (3.2) holds for  $l = 1$  and  $i_1 = 0$ .

<sup>4</sup>the speed of convergence would depend on the choice of  $x$ .

Assume that  $p_0 = a_0$ . Then, either (a)  $G_i(C_0) = p_i, i \in \Sigma_N$ , or (b)  $H_i(C_0) = C_0, i \in \Sigma_N$ , fails, because if both (a) and (b) hold, then, (4.12), (4.13) and (4.14) follow.

Assume (a) fails. Then, by using that  $p_0 = a_0$  and  $G_0$  is strictly increasing,  $\cap_i G_i^{-1}(p_i) = \emptyset$ . Since each  $G_i$  is continuous and  $Y$  is compact, (3.2) holds for any  $l$  and  $i_1, \dots, i_l$ . Assume (b) fails. Then,  $H_i(C_0) \neq C_0$  for some  $i$ , and (3.2) holds for  $l = 1$  and  $i_1 = i$ .  $\square$

**Example 4.9** (Linear case). If all  $c_i$  are zero, that is, all  $J_i$  are affine maps, then,  $\alpha = \beta = 0$ , and hence  $Y = \{0\}$  and  $G_i(0) = a_i$ . Then,  $\mu_f$  is the  $(a_0, \dots, a_{N-1})$ -self-similar measure. We have that

$$\dim_H \mu_f = \frac{s_N((a_0, \dots, a_{N-1}))}{\log N}.$$

**Example 4.10** ((B-y) fails but (sB-y) holds). Let  $N = 2$ . Consider (4.1) for

$$A_0 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix},$$

then,  $J_0(x) = x/(x+1)$  and  $J_1(x) = 1/(2-x)$ . Then, (A1)-(A3) holds and hence the solution of (4.1) exists.  $f$  is the inverse function of Minkowski's question-mark function [Mi1904]. But (sA3) fails. Then,  $Y = [-1, +\infty]$ ,

$$G_0(x) = \frac{x+1}{x+2}, \quad H_0(x) = x+1, \quad H_1(x) = -\frac{1}{x+2},$$

and,  $\mu_f = \mu_0$ . In this case, (A-y) may fail, but (1.3) holds for *any*  $i_1, i_2 \in \{0, 1\}$ . By Lemma 2.6, there is a constant  $\epsilon_1 > 0$  such that for any  $i \in \mathbb{N}$  and *any*  $x \in \{0, 1\}^{\mathbb{N}}$ ,

$$\epsilon_1 + s_2(p_i(y; x)) + s_2(p_{i+1}(y; x)) < 2 \log 2.$$

Hence by Theorem 1.3 (ii), we have that  $\dim_H \mu_f < 1$ . In this case (wA-y) fails for  $y = 0$ . We are not sure whether  $\dim_H \mu_f > 0$ . By Proposition 3.3,  $\mu_f$  is singular with respect to any self-similar measure.

**Remark 4.11.** If  $\tilde{\mu}$  is the measure on  $[0, 1]$  such that its distribution function is Minkowski's question-mark function, then, by [Kin60] it is known that

$$\dim_H \tilde{\mu} = \frac{\log 2}{2 \int_{[0,1]} \log(1+x) \tilde{\mu}(dx)} > \frac{1}{2}. \quad (4.15)$$

**Remark 4.12.** (i) In the case that each  $J_i$  is a linear fractional transformation, we can take  $Y$  as a subset of the set of real numbers. However, if some  $J_i$  are not linear fractional transformations, then, we may not be able to take  $Y$  as a subset of  $\mathbb{R}$ . Therefore, our approach may not work well, at least in a direct manner. One difficulty is that a set of functions can be very large, informally speaking.

(ii) The approaches in [Ha85, SLK04, Ok16] are different from the one used here. As an outline level, they are somewhat similar to each other.

## 5. OPEN PROBLEMS

(1) Give estimates for the upper and lower local dimensions of  $\mu_y$  and consider the Hausdorff dimensions for the level sets. Consider other notions of dimensions for  $\mu_y$ , such as  $L^p$ -dimensions. See [St93].

(2) If  $Z = [0, 1]$ , then, under what conditions is  $\mu_y$  a Rajchman measure [R28], that is, a measure whose Fourier transform vanishes at infinity? See [Ly95] for more

information of Rajchman measures. Recently, [JS16] shows that the Fourier coefficients of  $\tilde{\mu}$  decay to zero, by considering conditions which assure a given measure invariant with respect to the Gauss map is a Rajchman measure, and using (4.15).

(3) Let  $Z = [0, 1]$ . It would be interesting to consider structures of  $L^2([0, 1], \mu_y)$ . For example, find a subset  $P$  of  $\mathbb{N}$  such that  $\{\exp(2\pi i k) : k \in P\}$  forms an orthonormal basis of  $L^2([0, 1], \mu_y)$ . [JP98] considers this question for a class of self-similar measures.

*Acknowledgment.* The author was supported by Grant-in-Aid for JSPS Fellows (16J04213).

## REFERENCES

- [ADF14] E. de Amo, M. Díaz Carrillo, and J. Fernández Sánchez, Harmonic analysis on the Sierpiński gasket and singular functions. *Acta Math. Hungar.* 143 (2014) 58-74.
- [A67] K. Azuma, Weighted sums of certain dependent random variables. *Tôhoku Math. J.* (2) 19 (1967) 357-367.
- [BKK] B. Bárány, G. Kiss, I. Kolossváry, Pointwise regularity of parameterized affine zipper fractal curves, arXiv:1608.04558.
- [BST99] O. Ben-Bassat, R.S. Strichartz, and A. Teplyaev, What is not in the domain of the Laplacian on Sierpiński gasket type fractals. *J. Funct. Anal.* 166 (1999) 197-217
- [BK00] L. Berg and M. Krüppel, De Rham's singular function and related functions. *Z. Anal. Anwendungen* 19 (2000) 227-237.
- [DL91] I. Daubechies and J. C. Lagarias, Two-scale difference equations. I. Existence and global regularity of solutions. *SIAM J. Math. Anal.* 22 (1991) 1388-1410.
- [DL92-1] I. Daubechies and J. C. Lagarias, Two-scale difference equations. II. Local regularity, infinite products of matrices and fractals. *SIAM J. Math. Anal.* 23 (1992) 1031-1079.
- [DL92-2] I. Daubechies and J. C. Lagarias, Sets of matrices all infinite products of which converge. *Linear Algebra Appl.* 161 (1992), 227-263.
- [DS76] L. E. Dubins, and L. J. Savage, Inequalities for stochastic processes (how to gamble if you must). Corrected republication of the 1965 edition. Dover Publications, Inc., New York, 1976. 255 pp.
- [DMS98] S. Dubuc, J.-L. Merrien, and P. Sablonniere, The length of the de Rham curve, *J. Math. Anal. Appl.* 223:1 (1998), 182-195.
- [E93] G. Edgar, *Classics on Fractals*, Addison-Wesley, Reading, MA, 1993.
- [EM92] G. A. Edgar, and R. D. Mauldin, Multifractal decompositions of digraph recursive fractals. *Proc. London Math. Soc.* (3) 65 (1992), no. 3, 604-628.
- [ES16] B. Espinoza and R. A. Sáenz, Restrictions of harmonic functions and Dirichlet eigenfunctions of the Hata set to the interval. *Analysis (Berlin)* 36 (2016), no. 3, 135-146.
- [F97] K. Falconer, *Techniques in fractal geometry*, John Wiley and Sons, 1997.
- [G93] R. Girgensohn, Functional equations and nowhere differentiable functions, *Aequationes Math.* 46 (1993) 243-256.
- [Ha85] M. Hata, On the structure of self-similar sets, *Japan J. Appl. Math.* 2 (1985) 381-414.
- [Hi04] M. Hino, On singularity of energy measures on self-similar sets, *Probab. Theory Relat. Fields* 132 (2005) 265-290.
- [Hu81] J. E. Hutchinson, Fractals and self-similarity, *Indiana Univ. Math. J.* 30 (1981) 713-747.
- [JOP] A. Johansson, A. Öberg, M. Pollicott, Ergodic Theory of Kusuoka Measures, available at arXiv:1506.03037v3.
- [JS16] T. Jordan, and T. Sahlsten, Fourier transforms of Gibbs measures for the Gauss map. *Math. Ann.* 364 (2016) 983-1023.
- [JP98] P. E. T. Jorgensen and S. Pedersen, Dense analytic subspaces in fractal  $L^2$ -spaces. *J. Anal. Math.* 75 (1998) 185-228.
- [Kak48] S. Kakutani, On equivalence of infinite product measures, *Ann. of Math.*, 49 (1948) 214-224.
- [Kaw11] K. Kawamura, On the set of points where Lebesgue's singular function has the derivative zero. *Proc. Jpn. Acad. Ser. A Math Sci.* 87, (2011) 162-166 .

- [Kig95] J. Kigami, Hausdorff dimensions of self-similar sets and shortest path metrics, *J. Math. Soc. Japan* 47 (1995) 381-404.
- [Kin60] J. R. Kinney, Note on a singular function of Minkowski. *Proc. Amer. Math. Soc.*, 11 (1960) 788-794.
- [Kr09] M. Krüppel, De Rham's singular function, its partial derivatives with respect to the parameter and binary digit sums. *Rostock. Math. Kolloq.* 64 (2009) 57-74.
- [Ku89] S. Kusuoka, Dirichlet forms on fractals and products of random matrices. *Publ. Res. Inst. Math. Sci.* 25 (1989) 659-680.
- [La73] P. D. Lax, The differentiability of Polya's function, *Adv. Math.*, 10 (1973) 456-464.
- [Ly95] R. Lyons, Seventy years for Rajchman measures, Proceedings of the Conference in Honor of Jean-Pierre Kahane, (Orsey 1993) *J. Fourier Anal. Appl.* Special Issue 1995 363-377.
- [Me98] J.-L. Merrien, Prescribing the length of a de Rham curve, *Math. Engrg. Indust.* 7:2 (1998), 129-138.
- [Mi1904] H. Minkowski, Verhandlungen des III Internationalen Mathematiker-Kongresses in Heidelberg, Berlin, 1904; also in *Gesammelte Abhandlungen*, vol. 2, 1991, pp. 50-51.
- [N04] P. Nikitin, The hausdorff dimension of the harmonic measure on de rham's curve. *Journal of Mathematical Sciences*, 121(3) (2004) 2409-2418.
- [Ok14] K. Okamura, Singularity results for functional equations driven by linear fractional transformations. *J. Theoret. Probab.* 27 (2014) 1316-1328.
- [Ok16] K. Okamura, On regularity for de Rham's functional equations. *Aequationes Math.* 90 (2016) 1071-1085.
- [Ol94] L. Olsen, Random geometrically graph directed self-similar multifractals, Pitman Research Notes in Mathematics Series, 307. Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, 1994.
- [P04] V. Y. Protasov. On the regularity of de rham curves. *Izvestiya: Mathematics*, 68(3):567, 2004.
- [P06] V. Y. Protasov. Fractal curves and wavelets. *Izvestiya: Mathematics*, 70(5):975, 2006.
- [PV17] V. Y. Protasov and A. S. Voynov. Matrix semigroups with constant spectral radius. *Linear Algebra Appl.*, 513: 376-408, 2017.
- [R28] A. Rajchman, Sur une classe de fonctions à variation bornée, *C. R. Acad. Sci. Paris*, 187, (1928) 1026-1028.
- [dR56] G. de Rham, Sur une courbe plane, *J. Math. Pures Appl.* 35 (1956) 25-42.
- [dR57] G. de Rham, Sur quelques courbes définies par des équations fonctionnelles, *Univ. e Politec. Torino. Rend. Sem. Mat.* 16 (1957), 101-113.
- [SB17] C. Serpa and J. Buescu Constructive Solutions for Systems of Iterative Functional Equations, *Constr. Approx.* 45 (2017) 273299.
- [SLK04] P. Shanga, X. Lib, S. Kamae, Fractal properties of de Rham-type curve associated to random walk, *Chaos, Solitons & Fractals* 21 (2004) 695-700
- [Sh80] A. N. Shiryaev, Probability. Translated from the first (1980) Russian edition by R. P. Boas, second ed. Graduate Texts in Mathematics 95, Springer-Verlag, New York, 1996
- [St93] R. S. Strichartz, Self-similar measures and their Fourier transforms. III. *Indiana Univ. Math. J.* 42 (1993), no. 2, 367-411.
- [TGD98] S. Tasaki, T. Gilbert, and J. R. Dorfman, An analytical construction of the SRB measures for Baker-type maps, *Chaos* 8 (1998), no. 2, 424-443.
- [Z01] M. C. Zdun, On conjugacy of some systems of functions. *Aequationes Math.* 61 (2001) 239-254.

RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES, KYOTO UNIVERSITY, KYOTO, 606-8502, JAPAN.

E-mail address: kazukio@kurims.kyoto-u.ac.jp